

Negational Fragment of Intuitionistic Control Logic

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Abstract

We investigate properties of monadic purely negational fragment of Intuitionistic Control Logic (ICL). This logic arises from Intuitionistic Propositional Logic (IPL) by extending language of IPL by additional new constant for *falsum*. Having two different falsum constants enables to define two forms of negation. We analyse implicational relations between negational monadic formulae and present a poset of non equivalent formulae of this fragment of ICL.

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1 Introduction

Intuitionistic Control Logic (ICL) was defined semantically and proof theoretically by Chuck Liang and Dale Miller in their joint work [1]. This logic can be seen as a combination of classical and intuitionistic logics. The original impetus for ICL came from the search for a logic that would preserve the crucial connective of intuitionistic implication and at the same time would be able to type programming language control operators such as *call/cc*. ICL adds to the language of IPL a new constant \perp which is distinct from intuitionistic falsum 0 . Having these two falsum constants 0 and \perp enables to define two forms of negation: $\sim A = A \rightarrow 0$ and $\neg A = A \rightarrow \perp$ respectively.

Let us compare negations in classical and intuitionistic logics. We denote intuitionistic negation by $\sim A$ and classical by $\neg A$. It corresponds with notation in ICL and intuitive meaning of $\neg A$ as “classical” negation in this logic. In Classical Propositional Logic (CPL) there exist only two non equivalent negational formulae: $A, \neg A$. The classical negation is involutive i.e. $\neg\neg A \leftrightarrow A$ is a CPL tautology, so it is not possible to define a new operator by iterating classical negation. In IPL there are three non equivalent

negational formulae: $A, \sim A, \sim\sim A$. It is known that $\sim\sim A$ does not imply A in intuitionistic logic, but $\sim\sim\sim A \leftrightarrow \sim A$ is an intuitionistic tautology. Thus using intuitionistic implication we can obtain a new operation which is a double negation $\sim\sim$. Further multiplying of intuitionistic negations will only give us (up to equivalence) a formula $\sim A$ in case of odd number of negations or a formula $\sim\sim A$ if there is even number of negations to start with.

In Intuitionistic Control Logic there are two distinct negations: $\sim A$ which is an ordinary intuitionistic negation and $\neg A$ which bears some characteristics of classical negation. Combination of these two kinds of negation results in possibility of forming new operators. We extract them and present their relations with respect to intuitionistic implication.

2 Preliminaries

In this section we recall some facts about ICL from [1]. We consider only propositional logic. Language of ICL consists of countably many variables denoted p_1, p_2, p_3, \dots , intuitionistic connectives $\vee, \wedge, \rightarrow$ called respectively disjunction, conjunction and implication and of three constants $0, 1, \perp$. As a shorthand for $(A \rightarrow B) \wedge (B \rightarrow A)$ we use the expression $A \leftrightarrow B$. A Kripke model for ICL will be called an *r-model* and is defined as follows.

Definition 1. A Kripke *r-model* is a quadruple of the form $\langle W, \mathbf{r}, \leq, \Vdash \rangle$ where W is a finite, non-empty set, \leq is a reflexive and transitive relation on the set W and \Vdash is a binary relation between elements of W and atomic formulae called *forcing*. Elements of the set W are called *worlds* or *nodes*. The element $\mathbf{r} \in W$ is the *root* of the model. It is the least element of the set W ($\mathbf{r} \leq u$ for every world $u \in W$).

The forcing relation \Vdash is monotone, that is if $u \leq v$ then $u \Vdash p$ implies $v \Vdash p$. The \Vdash relation is extended to all formulae in the following way. Let $u, v, i \in W$.

- $u \Vdash 1$ and $u \not\Vdash 0$
- $\mathbf{r} \not\Vdash \perp$
- $i \Vdash \perp$ for all $i > \mathbf{r}$
- $u \Vdash A \vee B$ iff $u \Vdash A$ or $u \Vdash B$
- $u \Vdash A \wedge B$ iff $u \Vdash A$ and $u \Vdash B$
- $u \Vdash A \rightarrow B$ iff for all $v \geq u$ if $v \Vdash A$ then $v \Vdash B$.

If a formula is forced in every world of an r-model, we say that it is satisfied in this r-model. If a formula A is satisfied in all r-models we say that it is valid, in symbols $\models A$.

Constants 0 and 1 corresponds to intuitionistic falsum and verum. A formula of ICL that does not contain constant \perp is an intuitionistic formula. Forcing of \perp distinguishes between the root of an r-model and the rest of worlds. We will call every world properly above the root an *imaginary world*. We use symbols u, v, w to represent arbitrary worlds in W and the symbol i to represent an imaginary world.

Because of two different constants for falsum, it is possible to define two different negations

Intuitionistic negation: $\sim p = p \rightarrow 0$

Classical negation: $\neg p = p \rightarrow \perp$

The term *classical* in the name of the second negation refers to the law of excluded middle which, with respect to this negation, is an ICL tautology. Let us suppose that $p \vee \neg p$ is refuted in the root of some r-model:

$$\mathbf{r} \not\models p \vee \neg p.$$

This is equivalent to

$$\mathbf{r} \not\models p \text{ and } \mathbf{r} \not\models \neg p$$

which implies that $\mathbf{r} \not\models p$ and there exists a world $u \geq \mathbf{r}$ such that $u \models p$ and $u \not\models \perp$. Condition $u \not\models \perp$ means that $u = \mathbf{r}$. Hence we get a contradiction. Nevertheless this negation is not fully classical — it is not involutive as $\neg\neg p$ does not imply p . It is because this negation is defined using intuitionistic implication. For this reason we prefer to call it \perp -negation, instead of "classical" negation.

In [1] Liang and Miller defined sequent calculus LJC for Intuitionistic Control Logic and proved soundness and completeness of LJC with respect to the Kripke semantics. However, in this paper we focus on the semantic approach and the equivalence between provability of a formula and its validity in all r-models is alluded to only in Theorem 4. The symbol $\vdash A$ denotes provability of a formula A in LJC.

For a background in intuitionistic logic see [2].

3 Negational fragment

In this paper we will consider monadic purely negational fragment of ICL, i.e. the fragment in the language of \sim , \neg and p only. It means that we treat both negations as primitive connectives, not defined by means of constants and implication. Formulae of this fragment will be called *n-formulae*.

It will be understood that N_k and N_m are different sequences of both negations and that $k, m \in \{0, 1, 2, \dots\} = \mathbb{N}$. To discriminate sequences of the same length we will use superscripts N_k^1, N_k^2 etc. By \sim^n and \neg^n we will understand iteration of n negations of given kind. We will denote by $N_k p$ an n -formula with k negations of both kinds. By *the length of an n -formula* $N_k p$ we define the number k of negations. Formulae of the form $N_{2j} p$ and $N_{2j+1} p$ will be called *even n -formula* and *odd n -formula*, respectively. We will treat the variable p as a negational formula of the length 0.

Every r -model defined as in Definition 1 is a model for monadic purely negational fragment of ICL as well. However, since negations \sim and \neg are our primitive notions, considering r -models for negational fragment we should define interpretation of these connectives independently.

Definition 2. A Kripke model for the negational fragment of ICL is a tuple $\mathcal{M} = \langle W, \mathbf{r}, \leq, \Vdash \rangle$ where W, r and \leq are defined as in Definition 1 and the forcing relation \Vdash is restricted to the variable p and constants $0, \perp$. Additionally we define the interpretation of negations:

- $u \Vdash \sim A$ iff $w \not\Vdash A$, for all $w \geq u$
- $u \Vdash \neg A$ iff $w \not\Vdash A$ or $w > \mathbf{r}$, for all $w \geq u$.

It is according to the definition of forcing for constants $0, \perp$ and intuitionistic implication in the case of full language. Forcing of intuitionistic negation is standard. For \perp -negation we have

$$u \Vdash \neg p \text{ iff } w \not\Vdash p \text{ or } w \Vdash \perp, \text{ for all } w \geq u.$$

The condition $w \Vdash \perp$ means that w is an imaginary world.

Fact 3. For every n -formula A we have:

1. $\mathbf{r} \Vdash \neg A$ iff $\mathbf{r} \not\Vdash A$,
2. $\mathbf{r} \not\Vdash \neg A$ iff $\mathbf{r} \Vdash A$,
3. $u \not\Vdash \neg A$ iff $u = \mathbf{r}$ and $u \Vdash A$ for arbitrary $u \in W$,
4. $i \Vdash \neg A$ for all $i > \mathbf{r}$.

The first point is straightforward from definition. In ICL the distinction between the root of the r -model and other worlds is expressed by the forcing of \perp , whereas in the monadic purely negational fragment the root of the model is the only world in which \perp -negation of a formula can be refuted:

$$u \not\Vdash \neg A \text{ iff } w \Vdash A \text{ and } w = \mathbf{r}, \text{ for some } w \geq u.$$

It follows that \perp -negation of a formula is forced in every imaginary world.

We are interested in relations between n-formulae and we investigate validity of formulae of the form $N_k p \rightarrow N_m p$. We denote by \mathcal{N} the set of all n-formulae. In the standard way we define an equivalence relation \equiv on the set \mathcal{N} :

$$A \equiv B \text{ iff } \models A \rightarrow B \text{ and } \models B \rightarrow A.$$

As usual, we consider the quotient set:

$$\mathcal{N}/\equiv = \{[A]_\equiv \mid A \in \mathcal{N}\},$$

where $[A]_\equiv$ is the equivalence class of a formula A . The relation \preceq on \mathcal{N}/\equiv is given by:

$$[A]_\equiv \preceq [B]_\equiv \text{ iff } \models A \rightarrow B.$$

Although the relation \preceq is defined on equivalence classes, no confusion should arise if we use it to denote a relation between two n-formulae:

$$A \preceq B \text{ iff } \models A \rightarrow B.$$

If for an n-formula $A = N_k p$ exists an n-formula $B = N_m p$ such that $A \equiv B$ and $m < k$, we say that A is *reducible* to B . In the other case we say that an n-formula A is *irreducible*.

Most proofs of facts about implicational relations between n-formulae are reduced to showing a contradiction in the procedure of finding a countermodel for a formula $A \rightarrow B$. For n-formulae of a length greater than 4 we repeatedly refer to extensionality.

Theorem 4. *For any formula $A(\bar{p}, s)$ and for all formulae B, C if $\vdash B \leftrightarrow C$ then*

$$\vdash A(\bar{p}, B/s) \leftrightarrow A(\bar{p}, C/s).$$

It is well-known that this theorem holds for IPL. The proof is by induction on the complexity of formulae. However, the case of additional constant \perp does not interfere with the proof, thus the theorem holds for ICL as well.

4 Relations between models

In a Kripke model for either ICL or its negational fragment let \circ mark a node in which a variable p is refuted and \bullet a node in which p is forced.

Firstly, let us consider two basic n-formulae $\sim p$ and $\neg p$. It is easy to see that minimal model and countermodel for $\sim p$ are \circ and \bullet respectively. In case of $\neg p$ due to Fact 3 we have $\mathbf{r} \Vdash \neg p$ iff $\mathbf{r} \not\Vdash p$ and $\mathbf{r} \not\Vdash \neg p$ iff $\mathbf{r} \Vdash p$. Thus a minimal model for $\neg p$ is \circ and minimal countermodel is \bullet .

The fact that $\sim p$ and $\neg p$ have the same minimal models and counter-models does not imply that these formulae are equivalent. There is a model in which $\neg p$ is satisfied and $\sim p$ is refuted, namely:



It is easy to see, that for every n-formula $N_k p$ the minimal model or countermodel are \circ and \bullet . These cases may seem not very interesting as they collapse both negations to the situation of ordinary classical negation. However, looking for a countermodel for an intuitionistic implication of two formulae is equivalent to looking for a model for the antecedent and a countermodel for the consequent. While considering an implication of n-formulae, one of these cases are frequently reduced to either \circ or \bullet , so it is sufficient to know if the variable can be forced or refuted in a given world of the model. This depends on the evenness of the sequence of negations preceding the variable. The following fact becomes useful in such situations.

Proposition 5. *For every world u in an r -model \mathcal{M} we have:*

1. *if $u \Vdash N_{2k} p$ then $w \Vdash p$ or $w > \mathbf{r}$, for some $w \geq u$,*
2. *if $u \Vdash N_{2k+1} p$ then $w \nVdash p$ or $w > \mathbf{r}$, for some $w \geq u$,*
3. *if $u \nVdash N_{2k} p$ then $w \nVdash p$ or $w > \mathbf{r}$, for some $w \geq u$,*
4. *if $u \nVdash N_{2k+1} p$ then $w \Vdash p$ or $w > \mathbf{r}$, for some $w \geq u$.*

Proof. We prove only one of the two most complex cases which is 2, others can be proven in an analogous way.

Let \mathcal{M} be an r -model and u an arbitrary world in this model. The proof is by induction on k .

Let $k = 0$ and let $A = N_1 p$. We have to consider two cases:

Case 1. $A = \sim p$.

Assume that $u \Vdash \sim p$. Then for all $w \geq u$ we have $w \nVdash p$ and the claim trivially follows.

Case 2. $A = \neg p$.

From the assumption that $u \Vdash \neg p$ it follows that either $u = \mathbf{r}$ and $u \nVdash p$ or $u > \mathbf{r}$. In both cases the claim follows.

For the induction step, let $k > 0$ and let $A = N_{2(k+1)+1} p$. Now, we consider following cases:

Case 1. $A = \sim \sim N_{2k+1} p$.

Assume that

$$u \Vdash \sim \sim N_{2k+1} p.$$

Then for every $v \geq u$ there exists a world $v' \geq v$ such that $v' \Vdash N_{2k+1}p$. Hence, by the induction hypothesis, there is $w \geq v'$ with

$$w \nVdash p \text{ or } w > \mathbf{r}.$$

In particular it follows that the claim holds.

Case 2. $A = \sim \neg N_{2k+1}p$.

From the assumption

$$u \Vdash \sim \neg N_{2k+1}p$$

we get that for every world $u' \geq u$ we have $u' \nVdash \neg N_{2k+1}p$. From point 3 of Fact 3 it follows that \mathbf{r} is the only world of the model and

$$\mathbf{r} \Vdash N_{2k+1}p,$$

which, by the induction hypothesis, obviously implies the claim.

Case 3. $A = \neg \sim N_{2k+1}p$.

Then from

$$u \Vdash \neg \sim N_{2k+1}p$$

it follows that, in particular,

$$u \nVdash \sim N_{2k+1}p \text{ or } u > \mathbf{r}.$$

If the latter holds, we are done. The former implies that there is $u' \geq u$ with

$$u' \Vdash N_{2k+1}p$$

and the claim follows from the induction hypothesis.

Case 4. $A = \neg \neg N_{2k+1}p$.

Assume that

$$u \Vdash \neg \neg N_{2k+1}p.$$

Then it follows that either $u > \mathbf{r}$, in which case the claim holds, or $u = \mathbf{r}$ and $u \nVdash \neg N_{2k+1}p$. If so, from Fact 3 we have $\mathbf{r} \Vdash N_{2k+1}p$ and again, by the induction hypothesis, we get the claim. \square

Let us consider an implication of two different n-formulae $N_k p \rightarrow N_m p$. Such a formula is never valid if the evenness of k and m is not the same. Indeed, suppose that

$$\mathbf{r} \nVdash N_k p \rightarrow N_m p$$

and let $k = 2j_1, m = 2j_2 + 1$ for some $j_1, j_2 \in \mathbb{N}$. Then there exists a world $u \geq \mathbf{r}$ such that

$$u \Vdash N_{2j_1} p \text{ and } u \nVdash N_{2j_2+1} p.$$

The countermodel for such a formula is \bullet . The other case of evenness is symmetrical with a countermodel \circ .

In fact for an implication of two different n-formulae we never need a countermodel of height greater than 2. That is because the implication which bounds two n-formulae is the only connective that in building a countermodel requires creating a new world possibly above the root. The search for a countermodel for a formula

$$A = N_k p \rightarrow N_m p$$

always starts with the assumption that $\mathbf{r} \not\models A$ which is equivalent to the fact that there is a world $u \geq \mathbf{r}$ such that

$$u \models N_k p \text{ and } u \not\models N_m p.$$

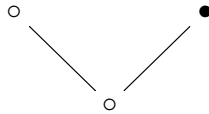
The minimal countermodel for an n-formula is either \circ or \bullet . Intuitionistic negation influences only the forcing of a subformula at the given world, regardless what kind of the world it is. Considering the \perp -negation of a formula requires discriminating the root from imaginary worlds. In other worlds, forcing of the variable in a given world is one of the two ways of distinguishing worlds in a model. The other one, as was already said, is refuting the \perp -negation of a formula.

Example 6. Let $A = \neg\neg\sim p$ and $B = \neg\sim\sim p$. We will show that $A \preceq B$ and $B \not\preceq A$.

Suppose that there exists an r-model in which $A \rightarrow B$ is not satisfied i.e. this formula is refuted in some world of this model. Showing a contradiction will give us $A \preceq B$.

Refutation of a formula in some world of an r-model means that it cannot be forced in the root of the model: $\mathbf{r} \not\models A \rightarrow B$, so there exists a world u possibly above the root, such that $u \models \neg\neg\sim p$ and $u \not\models \neg\sim\sim p$. The first condition says that either the world u is imaginary or $u = \mathbf{r}$ and the variable p is not forced in any node of the model. The latter condition implies that $u = \mathbf{r}$ and for every world w in the model exists $w' \geq w$ with $w' \models p$, hence a contradiction. It follows that $A \preceq B$.

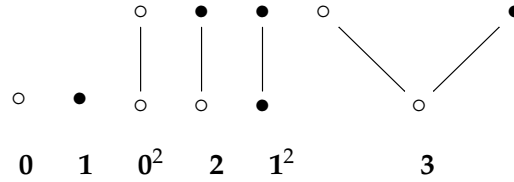
Suppose that for some r-model $\mathbf{r} \not\models B \rightarrow A$. Then again there exists a world u possibly above the root in which $u \models \neg\sim\sim p$ and $u \not\models \neg\neg\sim p$. Refuting the formula $\neg\neg\sim p$ in some world of a model means that this world is the root and there must exist a node somewhere in the model in which p is forced. If so, forcing of the formula $\neg\sim\sim p$ implies that there exists a world $w \geq \mathbf{r}$ for which $w' \not\models p$ for every $w' \geq w$, so in particular $\mathbf{r} \not\models p$. The least possible countermodel is:



The first part of the example shows that in looking for a countermodel the *kind* of a current world (either the root or imaginary) is important. The second part shows the difference of forcing of the variable in imaginary worlds. No bigger models would be necessary, as there is only one variable to validate and only one intuitionistic implication.

In other worlds forcing or refuting an intuitionistic negation of a formula in a world possibly above the root cannot extort creating a new world properly above. Forcing of \perp -negation of a formula in the root depends on the forcing of the variable, above the root it is always forced. The case of refuted \perp -negation of a formula sends us back to the root.

For a monadic negational formula there are two possible countermodels of the height of 1 and four possible countermodels of the height of 2:



Let us denote the set $\{0, 1, 0^2, 2, 1^2, 3\}$ of these models by \mathcal{S} .

We are interested in finding the upper bound of the number of non equivalent n-formulae. We are looking for n-formulae A and B such that $A \not\equiv B$. It means that there exists a model for the formula A in which we can refute the other n-formula. To every implication $A \rightarrow B$ we can assign a subset of \mathcal{S} of models in which this formula is refuted. Such subset cannot contain both of the models \circ and \bullet , because contradiction is not expressible in our language. There are 2^5 such subsets of \mathcal{S} , so there are at most 32 non equivalent monadic n-formulae.

For a given n-formula every model from the set \mathcal{S} can be either a model or a countermodel. The existence of only six possible models for monadic n-formulae enables to characterize these formulae in terms of their models. Let '+' stand for 'valid', and '-' for 'not valid'. For example for the formula p we have

N_0	0	1	0^2	2	1^2	3
p	-	+	-	-	+	-

and for n-formulae of the length 2 we have

N_2	0	1	0^2	2	1^2	3
$\sim\sim p$	-	+	-	+	+	-
$\sim\neg p$	-	+	-	-	-	-
$\neg\sim p$	-	+	-	+	+	+
$\neg\neg p$	-	+	-	-	+	-

It can be seen that there are two countermodels for a formula $\sim\sim p \rightarrow \sim\neg p$, namely **2** and 1^2 . However, we could also see that sets of models for p and $\neg\neg p$ are the same, and it is known, that \perp -negation is not involutive. Obviously some informations are missing.

In the case of IPL, when we look for a countermodel of a formula $A \rightarrow B$, we can always start with the assumption that it is already falsified in the root of the model, i.e. we can assume that $\mathbf{r} \Vdash A$ and $\mathbf{r} \nVdash B$. That is because in IPL the root of the model is no different from other worlds. We already saw that in ICL there is a considerable difference between the root and any imaginary world, e.g. the root is the only world of the model in which $\neg A$ can be refuted. It is not enough to look for countermodels of an n-formula starting in the root of an r-model. The forcing of it has to be also revised in those possible imaginary worlds of models $0^2, 2, 1^2$ and **3**. It is not sufficient to examine sets of models and countermodels for each n-formula. Its validity has to be also verified in every *pseudosubmodel*. Here by the pseudosubmodel we mean any generated submodel in the sense of IPL which is not an r-model, that is which consists imaginary worlds only. Let us denote $i(0), i(1)$ and $i(0, 1)$ pseudosubmodels of $0^2, 1^2, 3$ respectively. The pseudosubmodel of **2** is the same as that of 1^2 .

For p and $\neg\neg p$ we have

	$i(0)$	$i(1)$	$i(0, 1)$
p	-	+	+
$\neg\neg p$	+	+	+

Indeed, assume that

$$\mathbf{r} \nVdash \neg\neg p \rightarrow p.$$

Then in some world $u \geq \mathbf{r}$ we have $u \Vdash \neg\neg p$ and $u \nVdash p$. Forcing of a double \perp -negation of the variable in a world possibly above the root implies

that

either $u > \mathbf{r}$ or $u = \mathbf{r}$ and $u \Vdash p$,

hence the countermodel for a formula $\neg\neg p \rightarrow p$ is $\mathbf{0}^2$.

5 Semantic characterisation of n-formulae

It was already emphasised that no n-formula with odd number of negations can be equivalent to n-formula with even number of negations. Therefore all properties concerning equivalences between n-formulae are divided into two cases: for odd and even length of n-formulae. The property of reduction of negations with respect to sequences of one type of negation is straightforward and it follows from the fact that both $\sim\sim\sim p \equiv \sim p$ and $\neg\neg\neg p \equiv \neg p$.

Proposition 7. *For any $k \in \mathbb{N}$ we have:*

1. $\sim^{(2k+2)} p \equiv \sim\sim p$,
2. $\sim^{(2k+1)} p \equiv \sim p$,
3. $\neg^{(2k+2)} p \equiv \neg\neg p$,
4. $\neg^{(2k+1)} p \equiv \neg p$.

Proof. Ad 1 and 2

Equivalence between $\sim p$ and $\sim\sim\sim p$ is an intuitionistic tautology. From this and extensionality we get the thesis.

Ad 3 and 4

Let us see that $\neg\neg\neg p \equiv \neg p$. Suppose that $\not\models \neg\neg\neg p \rightarrow \neg p$, so there exists an r-model in which $\mathbf{r} \not\models \neg\neg\neg p \rightarrow \neg p$. Then there exists a world $u \geq \mathbf{r}$ such that

$$u \Vdash \neg\neg\neg p \text{ and } u \not\models \neg p.$$

According to point 3 of Fact 3, refuting the \perp -negation of a formula in some world of a model sends us back to the root, so the latter condition implies that $u = \mathbf{r}$ and $u \Vdash p$. If so, from the former condition and point 1 of Fact 3 it follows that in the root the formula $\neg\neg p$ is refuted. A contradiction, since this means that $\mathbf{r} \not\models p$.

Suppose that the formula $\neg p \rightarrow \neg\neg\neg p$ is not valid. Then in the root of some r-model we have $\mathbf{r} \not\models \neg p \rightarrow \neg\neg\neg p$. Hence for a world u possibly above the root we have

$$u \Vdash \neg p \text{ and } u \not\models \neg\neg\neg p.$$

Similar argumentation as in previous case with respect to the second condition shows that $u = \mathbf{r}$ and $u \Vdash \neg\neg p$. From points 1 and 2 of Fact 3 we

have that $\mathbf{r} \Vdash p$. On the other hand, since $u = \mathbf{r}$, from $u \Vdash \neg p$ follows $\mathbf{r} \nVdash p$, a contradiction. Therefore $\neg\neg p \equiv \neg p$ and the claim follows from extensionality. \square

In [1] Liang and Miller distinguished a formula $\sim\neg A \rightarrow A$. It enables to emulate the \mathcal{C} control operator. From point 1 of the following proposition it follows that the formula $\sim\neg p$ is a representative of a wide class of equivalent n-formulae of the form $\sim\neg N_{2k}p$. Point 2 shows a similar result for a class of odd n-formulae.

Proposition 8. *For any $k \in \mathbb{N}$ we have:*

1. $\sim\neg N_{2k}p \equiv \sim\neg p$,
2. $\sim\neg N_{2k+1}p \equiv \sim\neg\neg p$.

Proof. Ad 1

Assume there exists an r-model \mathcal{M} in which $\mathbf{r} \nVdash \sim\neg N_{2k}p \rightarrow \sim\neg p$. Thus there exists a world $u \geq \mathbf{r}$ for which we have

$$u \Vdash \sim\neg N_{2k}p \text{ and } u \nVdash \sim\neg p,$$

that is in all worlds above the world u the formula $\neg N_{2k}p$ is refuted, which means that the possible countermodel consists of the root only and that $\mathbf{r} \Vdash N_{2k}p$. Thus from point 1 of Proposition 5 we have $\mathbf{r} \Vdash p$. On the other hand, refuting the formula $\sim\neg p$ in the root in particular implies that $\mathbf{r} \nVdash p$. A contradiction.

Let \mathbf{r} be the root of some r-model in which a formula $\sim\neg p \rightarrow \sim\neg N_{2k}p$ is not satisfied. Then for some world $u \geq \mathbf{r}$ we claim that

$$u \Vdash \sim\neg p \text{ and } u \nVdash \sim\neg N_{2k}p.$$

As could be already seen in the case of reverse implication, forcing of the formula $\sim\neg p$ in arbitrary world implies that the model is reduced to the root and $\mathbf{r} \Vdash p$. From point 3 of Proposition 5 with respect to the second condition and the fact that the model comprises the root only, it yields that $\mathbf{r} \nVdash p$. From which we get a contradiction and as a result the claim follows.

Ad 2

Suppose that $\nVdash \sim\neg N_{2k+1}p \rightarrow \sim\neg\neg p$. So there exists an r-model such that for the root of it we have

$$\mathbf{r} \nVdash \sim\neg N_{2k+1}p \rightarrow \sim\neg\neg p$$

which implies that there exists a world $u \geq \mathbf{r}$ such that

$$u \Vdash \sim\neg N_{2k+1}p \text{ and } u \nVdash \sim\neg\neg p.$$

Similar argumentation as in the proof of point 1 and application of point 2 of Proposition 5 to the first condition implies that for all worlds $u' \geq u$ we have

$$u' = \mathbf{r} \text{ and } u' \not\models p.$$

If the root is the only world of the model, then in particular $u \not\models \sim \neg \neg p$ implies $u = \mathbf{r} \models p$, a contradiction.

Let us consider an r-model in which $\sim \neg \neg p \rightarrow \sim \neg N_{2k+1}p$ is not satisfied. Then this formula is refuted in the root of this model and there exists a world $u \geq \mathbf{r}$ such that

$$u \models \sim \neg \neg p \text{ and } u \not\models \sim \neg N_{2k+1}p.$$

The first condition implies that in every world $u' \geq u$ the formula $\neg \neg p$ is refuted and that means

$$u' = \mathbf{r} \text{ and } u' \not\models p,$$

whereas from the second condition follows that there is a world $v \geq u$ such that $v \models \neg N_{2k+1}p$. If so, by application of the point 4 of Proposition 5 and the fact of the root being the only world of the model, we have in particular

$$v' = \mathbf{r} \text{ and } v' \models p,$$

a contradiction. Thus the thesis holds. \square

Proposition 7 and Proposition 8 show that in many cases we can reduce an n-formula of a greater length to a formula of length less than 4.

In fact the formula $\sim \neg p$ implies every n-formula with even number of negations. That is because forcing of the formula $\sim \neg A$ at a given world sends us back to the root of the model in which A must be forced. Thus we get a minimal element with respect to relation \preceq for the subset of even n-formulae. Analogically the formula $\sim \neg \neg p$ is the minimal element for the subset of odd n-formulae. These two facts are corollaries from points 1 and 2 of Proposition 8 and the following proposition:

Proposition 9. *For every $k, m \in \mathbb{N}$ following implications hold:*

1. $\sim \neg N_{2k}p \rightarrow N_{2m}p$,
2. $\sim \neg N_{2k+1}p \rightarrow N_{2m+1}p$.

Proof. Ad 1

Assume $\mathbf{r} \not\models \sim \neg N_{2k}p \rightarrow N_{2m}p$ for the root of some r-model \mathcal{M} . Then in some world $u \geq \mathbf{r}$ we have $u \models \sim \neg N_{2k}p$ and $u \not\models N_{2m}p$. From the former it follows that the root is the only world of the model \mathcal{M} and $\mathbf{r} \not\models \neg N_{2k}p$. According to point 2 of Fact 3 it follows that $\mathbf{r} \models N_{2k}p$. On the other hand $\mathbf{r} \not\models N_{2m}p$, a contradiction.

Ad 2

Let \mathcal{M} be an r-model in which $\mathbf{r} \not\models \sim \neg N_{2k+1}p \rightarrow N_{2m+1}p$. Then there exists a world $u \geq \mathbf{r}$ such that $u \models \sim \neg N_{2k+1}p$ and $u \not\models N_{2m+1}p$. Hence, due to a similar reasoning as in point 1, since the root is the only world of the model we have $\mathbf{r} \models N_{2k+1}p$ and $\mathbf{r} \not\models N_{2m+1}p$, a contradiction. \square

It is worth noting that n-formulae $\sim \sim \neg \neg p$ and $\sim \sim \neg p$ are maximal elements with respect to the relation \preceq for subsets of even n-formulae and odd n-formulae, respectively.

Proposition 10. *For any $k \in \mathbb{N}$ following implications are valid:*

1. $N_{2k}p \rightarrow \sim \sim \neg \neg p$,
2. $N_{2k+1}p \rightarrow \sim \sim \neg p$.

Proof. Ad 1

Let \mathcal{M} be an r-model in which $\mathbf{r} \not\models N_{2k}p \rightarrow \sim \sim \neg \neg p$. Then there exist a world u possibly above the root such that $u \models N_{2k}p$ and $u \not\models \sim \sim \neg \neg p$. The latter implies that the model \mathcal{M} consists of only one element, namely the root and $\mathbf{r} \not\models p$. If there is no worlds properly above the root, then from $u \models N_{2k}p$, according to point 1 of Proposition 5 it follows in particular that $\mathbf{r} \models p$, a contradiction.

Ad 2

Suppose that there exists an r-model \mathcal{M} in which $\mathbf{r} \not\models N_{2k+1}p \rightarrow \sim \sim \neg p$. Then there exists a world $u \geq \mathbf{r}$ such that $u \models N_{2k+1}p$ and $u \not\models \sim \sim \neg p$. According to point 2 of Proposition 5 it follows that there exists a world $w \geq u$ such that $w \not\models p$ or $w > \mathbf{r}$. On the other hand, refuting the formula $\sim \sim \neg p$ in an arbitrary world of the model means that there are no worlds properly above the root and $\mathbf{r} \models p$. We have a contradiction and hence the claim holds. \square

For every $k \in \mathbb{N}$ there are 2^k n-formulae N_kp . It was already said that there are at most 32 non equivalent n-formulae. Procedure of finding these n-formulae is reduced to checking if the relation $N_kp \preceq N_mp$ holds. It would be arduous if it weren't for the fact that we can characterize a negational formula in terms of its models and countermodels. Instead of checking satisfiability of formulae of the form $N_kp \rightarrow N_mp$ for subsequent n-formulae N_kp, N_mp , it is sufficient to compare sets of models and countermodels, including pseudosubmodels, for these n-formulae. For a given n-formula N_kp let $\mathcal{S}^+(N_kp)$ be the subset of $\mathcal{S} \cup \{i(\mathbf{0}), i(\mathbf{1}), i(\mathbf{0}, \mathbf{1})\}$ of models in which n-formula N_kp is valid. The relation $N_kp \preceq N_mp$ between two n-formulae holds if and only if $\mathcal{S}^+(N_kp) \subseteq \mathcal{S}^+(N_mp)$. Complete tables of models for a given n-formula up to the length 5 are given in the Appendix.

Semantically all proofs of following facts are similar to the proof of Proposition 7. They are not informative, thus omitted.

As a representative for every equivalence class we choose a formula of the smallest length. We start with three equivalence classes of the simplest n-formulae, namely

$$[p]_{\equiv}, [\sim p]_{\equiv} \text{ and } [\neg p]_{\equiv}$$

and we will shortly discuss subsequent n-formulae.

Fact 11. *All n-formulae of length 2 are pairwise non equivalent.*

As none of n-formulae N_2p is reducible, we can distinguish four different equivalence classes:

$$[\sim\sim p]_{\equiv}, [\sim\neg p]_{\equiv}, [\neg\sim p]_{\equiv}, [\neg\neg p]_{\equiv}.$$

There are 8 n-formulae N_3p . From Proposition 7 it follows that $\sim\sim\sim p$ and $\neg\neg\neg p$ are reducible to $\sim p$ and $\neg p$ respectively.

Fact 12. *There are only two irreducible formulae N_3p that are not equivalent to any other n-formulae of the length 3, namely $\neg\sim\sim p$ and $\neg\neg\sim p$.*

Fact 13. *For n-formulae of the length 3 we have following equivalences:*

1. $\sim\sim\neg p \equiv \neg\sim\neg p,$
2. $\sim\neg\sim p \equiv \sim\neg\neg p.$

From Proposition 7.2, Proposition 7.4, Fact 12 and Fact 13 it follows that there are only 4 irreducible and not equivalent n-formulae N_3p . We choose following representatives:

$$[\sim\neg\neg p]_{\equiv}, [\neg\neg\sim p]_{\equiv}, [\neg\sim\sim p]_{\equiv} \text{ and } [\sim\sim\neg p]_{\equiv}.$$

Most of n-formulae N_4p could be reduced to some n-formula N_2p using Proposition 7 and Proposition 8. Remaining n-formulae are divided into three equivalence classes.

Fact 14. *For n-formulae of the length 4 we have following equivalences:*

1. $\sim\sim\neg\sim p \equiv \neg\neg\sim\sim p \equiv \sim\sim\neg\neg p \equiv \neg\sim\neg\neg p,$
2. $\neg\neg\sim\neg p \equiv \neg\sim\sim\neg p.$

Fact 15. *There is only one irreducible n-formula of length 4 that is not equivalent to any other n-formula N_4p , namely $\neg\neg\sim\sim p$.*

Recapitulating, we can distinguish three representatives of irreducible and not equivalent n-formulae of the length 4 which will denote equivalence classes:

$$[\neg\sim\sim\neg p]_{\equiv}, [\neg\neg\sim\sim p]_{\equiv} \text{ and } [\sim\sim\neg\neg p]_{\equiv}.$$

Fact 16. *There are only 4 irreducible n-formulae of the length 5. These formulae are equivalent: $\neg\sim\sim\neg\neg p \equiv \neg\sim\neg\neg p \equiv \neg\neg\sim\neg p \equiv \neg\neg\sim\neg p$.*

The equivalence class of these n-formulae will be denoted by

$$[\neg\sim\sim\neg\neg p]_{\equiv}.$$

All remaining n-formulae N_5p are reducible to some formulae N_3p . Equivalences between n-formulae N_5p and N_3p are based on Proposition 7, Proposition 8 and extensionality. There is only one not obvious case namely $\neg\neg\sim\sim\neg p \equiv \sim\sim\neg p$. Indeed, let us suppose that $\not\models \neg\neg\sim\sim\neg p \rightarrow \sim\sim\neg p$. Then there exists some model \mathcal{M} in which $\mathbf{r} \not\models \neg\neg\sim\sim\neg p \rightarrow \sim\sim\neg p$. That is there exists some world $u \geq \mathbf{r}$ such that $u \models \neg\neg\sim\sim\neg p$ and $u \not\models \sim\sim\neg p$. From the second condition follows that there exists a world $v \geq u$ such that for every world $w \geq v$ we have $w \not\models p$. This means that the world w is the root of the model \mathcal{M} and the only world of it and $\mathbf{r} \models p$. On the other hand we have $\mathbf{r} \models \neg\neg\sim\sim\neg p$ which implies that $\mathbf{r} \not\models p$, a contradiction. Let us suppose that $\not\models \sim\sim\neg p \rightarrow \neg\neg\sim\sim\neg p$. Then again for the root of some model \mathcal{M} we have $\mathbf{r} \not\models \sim\sim\neg p \rightarrow \neg\neg\sim\sim\neg p$. Hence there is a world $u \geq \mathbf{r}$ such that $u \models \sim\sim\neg p$ and $u \not\models \neg\neg\sim\sim\neg p$. The second part implies that $u = \mathbf{r}$ and $u \not\models \sim\sim\neg p$, a contradiction.

Corollary 17. *Every n-formula N_6p is reducible to some n-formula N_kp of length $k \leq 4$.*

Proof. For every n-formula N_6p there exists a sequence N_5 such that either $N_6p = \sim N_5p$ or $N_6p = \neg N_5p$. We consider two cases.

Case 1. The formula N_5p is reducible to some n-formula N_3p . Then we have either $N_6p \equiv \sim N_3p$ or $N_6p \equiv \neg N_3p$, which implies that every n-formula N_6p is reducible to some n-formula N_4p .

Case 2. The formula N_5p is irreducible. Then from Fact 16 follows that $N_5p \in [\neg\sim\sim\neg\neg p]_{\equiv}$. If $N_6p = \sim N_5p$ then in particular $N_6p \equiv \sim\neg\sim\sim\neg\neg p$ and from Proposition 8.1 follows that $N_6p \equiv \sim\neg p$. If $N_6p = \neg N_5p$ then in particular $N_6p \equiv \neg\neg\neg\sim\sim\neg\neg p$. From point 4 of Proposition 7 it follows that $N_6p \equiv \neg\sim\neg\neg p$. \square

Theorem 18. *Every n-formula N_kp , for $k \geq 6$ is reducible to some n-formula of length $m \leq 5$.*

Proof. By induction on k . The induction base follows from Corollary 17.

Assume that $k > 6$. Let $\diamond \in \{\sim, \neg\}$. Let $N_{k+1}p = \diamond N_kp$. From the induction hypothesis there exists a sequence N_m such that $m \leq 5$ and $N_kp \equiv N_mp$. Thus we have $N_{k+1}p \equiv \diamond N_mp$. If $m < 5$, we get the thesis. Else it is the case of the induction base. \square

6 Equivalence classes of n-formulae

As a conclusion from the previous section, especially Facts 11– 16, Corollary 17 and Theorem 18 we get the exact power of the set \mathcal{N}/\equiv .

Theorem 19. *There are exactly 15 (up to equivalence) pairwise not equivalent and irreducible n-formulae:*

$$\begin{array}{cc}
 p & \\
 \sim\sim p & \sim p \\
 \sim\neg p & \neg p \\
 \neg\sim p & \sim\sim\neg p \\
 \neg\neg p & \sim\neg\neg p \\
 \sim\sim\neg\neg p & \neg\sim\sim p \\
 \neg\sim\sim\neg p & \neg\neg\sim p \\
 \neg\neg\sim\sim p & \neg\sim\sim\neg p
 \end{array}$$

Two following theorems gather all relations between elements of the set \mathcal{N}/\equiv . We present them in the form of implications of n-formulae for chosen representatives of equivalence classes, as this is more readable. By stating that an implication $N_k p \rightarrow N_m p$ is valid we mean that the reverse implication is not valid.

Theorem 20. *Following implications of even n-formulae are valid:*

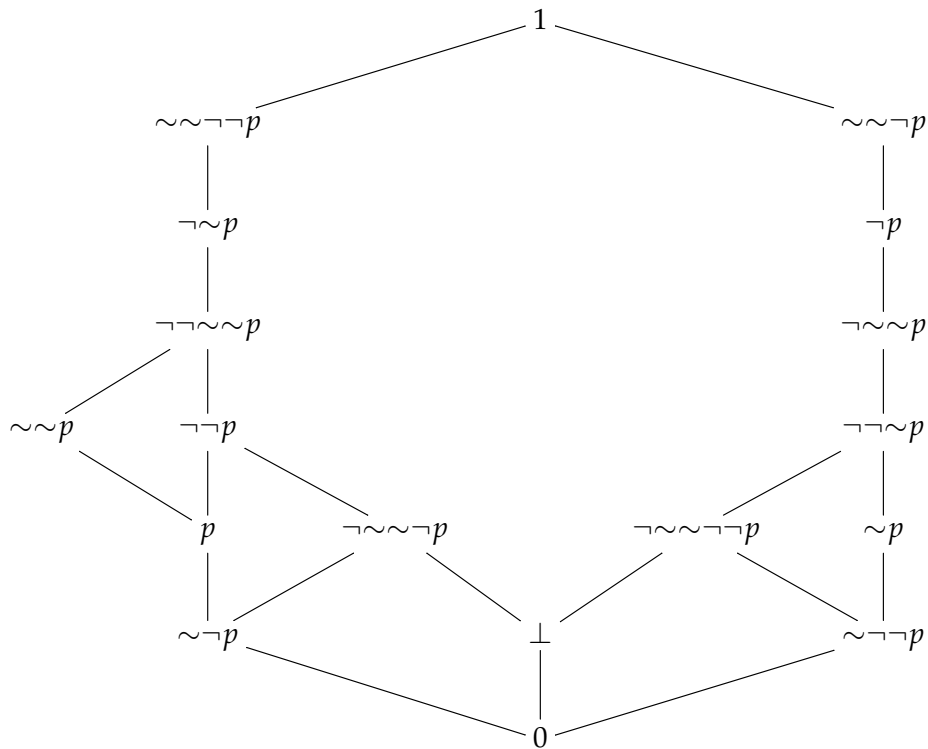
1. $\sim\neg p \rightarrow p$
2. $p \rightarrow \neg\neg p$
3. $\neg\neg p \rightarrow \neg\neg\sim\sim p$
4. $\neg\neg\sim\sim p \rightarrow \neg\sim p$
5. $\neg\sim p \rightarrow \sim\sim\neg\neg p$
6. $\sim\neg p \rightarrow \neg\sim\sim\neg p$
7. $\neg\sim\sim\neg p \rightarrow \neg\neg p$
8. $p \rightarrow \sim\sim p$
9. $\sim\sim p \rightarrow \neg\neg\sim\sim p$

Theorem 21. *Following implications of odd n -formulae are valid:*

1. $\sim \neg \neg p \rightarrow \sim p$
2. $\sim p \rightarrow \neg \neg \sim p$
3. $\neg \neg \sim p \rightarrow \neg \neg \neg p$
4. $\neg \neg \neg p \rightarrow \neg p$
5. $\neg p \rightarrow \neg \neg \neg p$
6. $\sim \neg \neg p \rightarrow \neg \neg \neg \neg p$
7. $\neg \neg \neg \neg p \rightarrow \neg \neg \sim p$

Proof. As a proof of both Theorem 20 and Theorem 21 we refer the Reader to validity tables in the Appendix. Checking the validity of a formula $N_k p \rightarrow N_m p$ (and stating that the reverse implication is not valid) is reduced to checking if $\mathcal{S}^+(N_k p) \subsetneq \mathcal{S}^+(N_m p)$. \square

Relations between classes of equivalent formulae are described by Lindenbaum Algebra. In the case of negational monadic fragment of ICL we cannot create such a structure. So we present a poset $(\mathcal{N}^* \cup \{0, \perp, 1\}, \preceq)$, where \mathcal{N}^* is the set of chosen representatives of equivalence classes of \mathcal{N}/\equiv .



The addition of constants enables to join the two posets of equivalence classes of even and odd n -formulae.

After looking into properties of negational formulae several questions arose. They consider mainly computational content of n -formulae. Further work would also include investigations of implicational fragment of ICL.

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Appendix

We give complete validity tables for n-formulae $N_k p$ up to the length of 5.

	0	1	0 ²	2	1 ²	3	$i(0)$	$i(1)$	$i(0,1)$
p	-	+	-	-	+	-	-	+	+
$\sim p$	+	-	+	-	-	-	+	-	-
$\neg p$	+	-	+	+	-	+	+	+	+
$\sim\sim p$	-	+	-	+	+	-	-	+	-
$\sim\neg p$	-	+	-	-	-	-	-	-	-
$\neg\sim p$	-	+	-	+	+	+	+	+	+
$\neg\neg p$	-	+	-	-	+	-	+	+	+
$\sim\sim\sim p$	+	-	+	-	-	-	+	-	-
$\sim\sim\neg p$	+	-	+	+	+	+	+	+	+
$\sim\neg\sim p$	+	-	-	-	-	-	-	-	-
$\sim\neg\neg p$	+	-	-	-	-	-	-	-	-
$\neg\sim\sim p$	+	-	+	-	-	+	+	+	+
$\neg\sim\neg p$	+	-	+	+	+	+	+	+	+
$\neg\neg\sim p$	+	-	+	-	-	-	+	+	+
$\neg\neg\neg p$	+	-	+	+	-	+	+	+	+
$\sim\sim\sim\sim p$	-	+	-	+	+	-	-	+	-
$\sim\sim\sim\neg p$	-	+	-	-	-	-	-	-	-
$\sim\sim\neg\sim p$	-	+	+	+	+	+	+	+	+
$\sim\sim\neg\neg p$	-	+	+	+	+	+	+	+	+

	0	1	0 ²	2	1 ²	3	$i(0)$	$i(1)$	$i(0,1)$
$\sim\sim\sim p$	-	+	-	-	-	-	-	-	-
$\sim\sim\sim p$	-	+	-	-	-	-	-	-	-
$\sim\sim\sim p$	-	+	-	-	-	-	-	-	-
$\sim\sim\sim p$	-	+	-	-	-	-	-	-	-
$\sim\sim\sim p$	-	+	-	+	+	+	+	+	+
$\sim\sim\sim p$	-	+	-	-	-	-	+	+	+
$\sim\sim\sim p$	-	+	+	+	+	+	+	+	+
$\sim\sim\sim p$	-	+	+	+	+	+	+	+	+
$\sim\sim\sim p$	-	+	-	+	+	-	+	+	+
$\sim\sim\sim p$	-	+	-	-	-	-	+	+	+
$\sim\sim\sim p$	-	+	-	+	+	+	+	+	+
$\sim\sim\sim p$	-	+	-	-	+	-	+	+	+
$\sim\sim\sim\sim p$	+	-	+	-	-	-	+	-	-
$\sim\sim\sim\sim p$	+	-	+	+	+	+	+	+	+
$\sim\sim\sim\sim p$	+	-	-	-	-	-	-	-	-
$\sim\sim\sim\sim p$	+	-	-	-	-	-	-	-	-
$\sim\sim\sim\sim p$	+	-	+	+	+	+	+	+	+
$\sim\sim\sim\sim p$	+	-	+	+	+	+	+	+	+
$\sim\sim\sim\sim p$	+	-	+	+	+	+	+	+	+
$\sim\sim\sim\sim p$	+	-	+	+	+	+	+	+	+
$\sim\sim\sim\sim p$	+	-	-	-	-	-	-	-	-
$\sim\sim\sim\sim p$	+	-	-	-	-	-	-	-	-

	0	1	0 ²	2	1 ²	3	<i>i</i> (0)	<i>i</i> (1)	<i>i</i> (0,1)
$\sim\sim\sim\sim p$	+	-	-	-	-	-	-	-	-
$\sim\sim\sim\sim p$	+	-	-	-	-	-	-	-	-
$\sim\sim\sim\sim p$	+	-	-	-	-	-	-	-	-
$\sim\sim\sim\sim p$	+	-	-	-	-	-	-	-	-
$\sim\sim\sim\sim p$	+	-	-	-	-	-	-	-	-
$\sim\sim\sim\sim p$	+	-	-	-	-	-	-	-	-
$\sim\sim\sim\sim p$	+	-	-	-	-	-	-	-	-
$\sim\sim\sim\sim p$	+	-	+	-	-	+	+	+	+
$\sim\sim\sim\sim p$	+	-	+	+	+	+	+	+	+
$\sim\sim\sim\sim p$	+	-	-	-	-	-	+	+	+
$\sim\sim\sim\sim p$	+	-	-	-	-	-	+	+	+
$\sim\sim\sim\sim p$	+	-	+	+	+	+	+	+	+
$\sim\sim\sim\sim p$	+	-	+	+	+	+	+	+	+
$\sim\sim\sim\sim p$	+	-	+	+	+	+	+	+	+
$\sim\sim\sim\sim p$	+	-	+	+	+	+	+	+	+
$\sim\sim\sim\sim p$	+	-	+	+	+	+	+	+	+
$\sim\sim\sim\sim p$	+	-	+	-	-	-	+	+	+
$\sim\sim\sim\sim p$	+	-	+	+	+	+	+	+	+
$\sim\sim\sim\sim p$	+	-	-	-	-	-	+	+	+
$\sim\sim\sim\sim p$	+	-	-	-	-	-	+	+	+
$\sim\sim\sim\sim p$	+	-	+	-	-	+	+	+	+
$\sim\sim\sim\sim p$	+	-	+	+	+	+	+	+	+
$\sim\sim\sim\sim p$	+	-	+	-	-	-	+	+	+
$\sim\sim\sim\sim p$	-	+	-	-	+	-	+	+	+